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S-INTEGRAL POINTS OF $\mathbb{P}^n - \{2n+1$ HYPERPLANES IN GENERAL POSITION$\}$ OVER NUMBER FIELDS AND FUNCTION FIELDS

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ABSTRACT. For the number field case we will give an upper bound on the number of the S-integral points in $\mathbb{P}^n(K) - \{2n+1$ hyperplanes in general position$\}$. The main tool here is the explicit upper bound of the number of solutions of S-unit equations (Invent. Math. **102** (1990), 95–107). For the function field case we will give a bound on the height of the S-integral points in $\mathbb{P}^n(K) - \{2n+1$ hyperplanes in general position$\}$. We will also give a bound for the number of "generators" of those S-integral points. The main tool here is the S-unit Theorem by Brownawell and Masser (Proc. Cambridge Philos. Soc. **100** (1986), 427–434).

1.0. INTRODUCTION—NUMBER FIELDS

Let K be a number field of degree d. Denote by M_K the set of valuations of K and by M_∞ the set of archimedean valuations of K. For each valuation $v \in M_K$, denote by K_v the completion of K with respect to v and by $n_v = [K_v : \mathbb{Q}_v]$ the local degree. Define an absolute value associated to an archimedean valuation v by

$$\|x\|_v = |x| \quad \text{if } K_v = \mathbb{R},$$
$$\|x\|_v = |x|^2 \quad \text{if } K_v = \mathbb{C}.$$

If v is non-archimedean, then v is an extension of a p-adic valuation on $\mathbb{Q}$ for some prime p; the absolute value is defined so that

$$\|x\|_v = |x|_p^{n_v}$$

if $x \in \mathbb{Q} - \{0\}$.

Let S be a finite subset of M_K containing the set M_∞. We call an element $x \in K$ an S-unit if

$$\|x\|_v = 1 \ \ \forall\ v \notin S.$$

Let D be a very ample effective divisor on a projective variety V and let $1 = x_0, x_1, ..., x_N$ be a basis of the vector space:

$$\mathcal{L}(D) = \{\ f | f \text{ is a rational function on the variety } V \text{ such that } f = 0 \text{ or } (f) + D \geq 0\}.$$

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Then $P \to (x_1(P), ..., x_N(P))$ defines an embedding of $V(K) - D$ into the affine space K^N. A point P of $V(K) - D$ is said to be an S-integral point if $\|x_i\|_v \leq 1 \ \forall\, v \notin S$ and for $1 \leq i \leq N$.

Recall that a hyperplane H in $\mathbb{P}^n(K)$ is represented by a vector α in $K^{n+1} - \{0\}$. A set of hyperplanes $\{H_1, ..., H_q\}$ is said to be in general position if the set of representing vectors $\{\alpha_1, ..., \alpha_q\}$ has the property that any subset of no more than $n+1$ elements is linearly independent over K.

Using Schmidt's Subspace Theorem and Nochka weights, Ru and Wong succesfully proved that the S-integral points of $\mathbb{P}^n(K) - \{2n+1$ hyperplanes in general position$\}$ are finite [R-W]. However, the bound of the number is not effectively determined. After Schmidt and Schlickewei succeeded in giving a quantitive version of the Subspace Theorem over number fields, Schlickewei obtained an upper bound for the number of solutions of S-unit equations [Schl]. This upper bound will be the main ingredient in our bound for the number of S-integral points in $\mathbb{P}^n(K) - \{2n+1$ hyperplanes in general position$\}$.

1.1. The Main Theorem—Number Fields

First we recall the following S-unit Theorem by Schlickewei [Schl]:

Theorem (Schlickewei). *Let K be a number field of degree d. Let $a_1, ..., a_n$ be nonzero elements of K. Suppose S is a finite subset of M_K of cardinality s, containing M_∞. Then the equation*

$$a_1 x_1 + \cdots + a_n x_n = 1 \tag{1.1}$$

has no more than

$$(4sd!)2^{36nd!s^6} \tag{1.2}$$

solutions in S-units $x_1, ..., x_n$ such that no proper subsum $a_{i_1}x_{i_1} + \cdots + a_{i_m}x_{i_m}$ vanishes.

Definition. Let $y_1, ..., y_m$ be elements in a field L and $y_1 + \cdots + y_m = 1$. Then $y_1 + \cdots + y_m = 1$ is an irreducible equation if no proper subsum $y_{i_1} + \cdots + y_{i_k}$ vanishes.

It is clear that we have the following proposition:

Proposition 1.1. *Let $y_1, ..., y_m$ be elements in a field L. If $y_1 + \cdots + y_m = 1$, then there exists a subsequence $y_{i_1}, ..., y_{i_k}$ of $y_1, ..., y_m$ such that $y_{i_1} + \cdots + y_{i_k} = 1$ is an irreducible equation.*

Remark. There may exist more than one irreducible equation with respect to $y_1 + \cdots + y_m = 1$. However the number of irreducible equations with respect to $y_1 + \cdots + y_m = 1$ is clearly finite.

Main Theorem (Number Fields). *Let K be a number field of degree d. Let $L_i = \sum_{j=0}^{n} a_{ij}x_j$, $1 \leq i \leq q$, be linear forms in $\mathbb{P}^n(K)$ and in general position. Denote by H_i the corresponding hyperplane of L_i. Suppose S is a finite subset of M_K of cardinality s, containing M_∞ and the smallest subset of nonarchimedean places v such that every nonzero coefficient a_{ij} of those q linear forms is an S-unit.*

If $q \geq 2n+1$, then the number of S-integral points in $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$ is bounded by

$$(4sd!)^n 2^{36n(n+1)d!s^6}. \tag{1.3}$$

Proposition 1.2. *Let S and the L_i, H_i be as in the Main Theorem. If $(x_0, ..., x_n)$ is an S-integral point in $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$, then*

$$\|L_i(x_0, ..., x_n)\|_v = \max_{0 \leq j \leq n} \{\|x_j\|_v\} \quad \forall\ 1 \leq i \leq q \text{ and } \forall\ v \notin S.$$

Proof. Let $X_0, ..., X_n$ be the coordinates of $\mathbb{P}^n(K)$. Then $\frac{X_0^q}{\prod_{i=1}^q L_i(\mathbf{X})}, \ldots, \frac{X_n^q}{\prod_{i=1}^q L_i(\mathbf{X})}$, are elements of $\mathcal{L}(\bigcup_{i=1}^q H_i)$ and are linearly independent. Therefore we can choose an embedding of $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$ into an affine space as follows:

$$(\frac{X_0^q}{\prod_{i=1}^q L_i(\mathbf{X})}, \ldots, \frac{X_n^q}{\prod_{i=1}^q L_i(\mathbf{X})}, \ldots). \tag{1.4}$$

Let $l_i := L_i(x_0, ..., x_n)$. If $(x_0, ..., x_n)$ is an S-integral point in $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$, then

$$\|\frac{x_j^q}{\prod_{i=1}^q l_i}\|_v \leq 1, \quad \forall\ v \notin S. \tag{1.5}$$

Therefore

$$\max_{0 \leq j \leq n} \{\|x_j\|_v\}^q \leq \prod_{i=1}^q \|l_i\|_v, \quad \forall\ v \notin S. \tag{1.6}$$

On the other hand, $l_i = \sum_{j=0}^n a_{ij} x_j$, where a_{ij}'s are S-units. Therefore

$$\|l_i\|_v \leq \max_{0 \leq j \leq n} \{\|x_j\|_v\}, \quad \forall\ v \notin S. \tag{1.7}$$

(1.6) and (1.7) imply that

$$\|l_i\|_v = \max_{0 \leq j \leq n} \{\|x_j\|_v\}, \quad \forall\ v \notin S \text{ and } \forall\ 1 \leq i \leq q. \tag{1.8}$$

This completes the proof.

Proposition 1.3. (a) *Let $L_1, ..., L_{n+1}$ be linear forms in $\mathbb{P}^n(K)$ and in general position. Let $(x_0, ..., x_n) \in \mathbb{P}^n(K)$ and $l_i = L_i(x_0, ..., x_n)$. Then there exist unique $b_1, ..., b_{n+1}$ in K such that $x_j = \sum_{i=1}^{n+1} b_i l_i$.*

(b) *Let $L_1, ..., L_{n+2}$ be linear forms in $\mathbb{P}^n(K)$ and in general position. Then there exist non-zero elements $D_1, ..., D_{n+2}$ in K such that*

$$D_1 L_1(\mathbf{X}) + \cdots + D_{n+2} L_{n+2}(\mathbf{X}) = 0. \tag{1.9}$$

Proof. (a) This is because that $L_1, ..., L_{n+1}$ are in general position.

(b) Suppose $L_i(\mathbf{X}) = \sum_{j=0}^n a_{ij} X_j,\ 1 \leq i \leq n+2$. Let

$$\triangle = \begin{pmatrix} a_{10} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n+2,0} & \cdots & a_{n+2,n} \end{pmatrix}.$$

Let $\triangle_i$ be the determinant of the matrix obtained by deleting the i-th row from $\triangle$. Since $L_1, ..., L_{n+2}$ are in general position, $\triangle_i \neq 0$ for $1 \leq i \leq n+2$. Since the determinant of the matrix obtained from inserting the column $(a_{1i}, ..., a_{n+2,i})^t$ to the matrix $\triangle$ is 0, we have the following equation:

$$0 = \begin{vmatrix} a_{1i} & a_{10} & \dots & a_{1i} & \dots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{n+2,i} & a_{n+2,0} & \dots & a_{n+2,i} & \dots & a_{n+2,n} \end{vmatrix} = a_{1i}\triangle_1 + (-1)a_{2i}\triangle_2 + \cdots + (-1)^{n+1}a_{n+2,i}\triangle_{n+2}. \tag{1.10}$$

Proof of the Main Theorem. If $(x_0, ..., x_n)$ is an S-integral point in $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$, then $\frac{l_i}{l_1}$ is an S-unit, for $1 \le i \le q$, where $l_i = L_i(x_0, ..., x_n)$. Let $I = \{i_0, ..., i_n\}$ be an index subset in $\{2, ..., q\}$. By Proposition 1.3.b, there exist non-zero elements $c_{i_j}^I$, $0 \le j \le n$, which depend only on the coefficients of $L_1, L_{i_0}, ..., L_{i_n}$ such that

$$L_1(\mathbf{X}) + c_{i_0}^I L_{i_0}(\mathbf{X}) + \cdots + c_{i_n}^I L_{i_n}(\mathbf{X}) = 0.$$

Therefore

$$1 + c_{i_0}^I \frac{l_{i_0}}{l_1} + \cdots + c_{i_n}^I \frac{l_{i_n}}{l_1} = 0. \tag{1.11}$$

If no proper subsum of (1.11) vanishes, then by the S-unit Theorem of number fields the number of solutions of $(\frac{l_{i_0}}{l_1}, \dots, \frac{l_{i_n}}{l_1})$ is less than

$$(4sd!)2^{36(n+1)d!s^6}.$$

Since the x_i can be written uniquely as a linear combination of $\{l_{i_0}, ..., l_{i_n}\}$, the $\frac{x_i}{l_1}$ can be written uniquely as a linear combination of $\{\frac{l_{i_0}}{l_1}, ..., \frac{l_{i_n}}{l_1}\}$. Therefore the number of S-integral points $(x_0, ..., x_n) = (\frac{x_0}{l_1}, ..., \frac{x_n}{l_1})$ in $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$ is no more than

$$(4sd!)2^{36(n+1)d!s^6}. \tag{1.12}$$

It then suffices to consider the case that for some index subset $I = \{i_0, ..., i_n\}$ in $\{2, ..., q\}$, some proper subsums of (1.11) vanish. By Proposition 1.1, there exist some irreducible equations in the following form:

$$-c_{j_0}^I \frac{l_{j_0}}{l_1} - \cdots - c_{j_k}^I \frac{l_{j_k}}{l_1} = 1, \tag{1.13}$$

where $I = \{i_0, ..., i_n\}$ is an index subset of $\{2, ..., q\}$ and $\{j_0, ..., j_k\}$ is an index subset of $\{i_0, ..., i_n\}$. There are only finitely many such irreducible equations in total. Let $T_{\mathbf{x}}$ be the subset of $\{L_1, ..., L_q\}$ with the following property:

(1.14)

$$L_i \in T_{\mathbf{x}} \text{ if and only if } \frac{l_i}{l_1} \text{ appears in one of the irreducible equations in (1.13).}$$

Rearranging the order of the linear forms, we may assume

$$T_{\mathbf{x}} = \{L_1, ..., L_u\}.$$

If $u < n+1$, by Proposition 1.3.b there exist c_i^I, $n+1 \le i \le 2n+1$, such that

$$L_1(\mathbf{X}) + c_{n+1}^I L_{n+1}(\mathbf{X}) + \cdots + c_{2n+1}^I L_{2n+1}(\mathbf{X}) = 0.$$

Since $L_{n+1}, ..., L_{2n+1} \notin T_{\mathbf{x}}$, no proper subsum of $c_{n+1}^I \frac{l_{n+1}}{l_1} + \cdots + c_{2n+1}^I \frac{l_{2n+1}}{l_1}$ vanishes. This is the previous case, which has already been treated.

If $u = n+1$, then $L_{n+2}, ..., L_{2n+1} \notin T_{\mathbf{x}}$. Again we have the following equations:

$$L_1(\mathbf{X}) + c_i^{I_i} L_i(\mathbf{X}) + c_{n+2}^{I_i} L_{n+2}(\mathbf{X}) + \cdots + c_{2n+1}^{I_i} L_{2n+1}(\mathbf{X}) = 0, \quad 2 \le i \le n+1.$$

Since $L_{n+2}, ..., L_{2n+1} \notin T_{\mathbf{x}}$ and we only need to consider the case that some proper subsum of

$$-c_i^{I_i} \frac{l_i}{l_1} - c_{n+2}^{I_i} \frac{l_{n+2}}{l_1} - \cdots - c_{2n+1}^{I_i} \frac{l_{2n+1}}{l_1} = 1, \quad 2 \le i \le n+1, \tag{1.15}$$

vanishes, the only possible irreducible equations for (1.15) are

$$-c_i^{I_i} \frac{l_i}{l_1} = 1, \quad 2 \le i \le n+1. \tag{1.16}$$

Therefore

$$\frac{l_i}{l_1} = -\frac{1}{c_i^{I_i}}, \quad 2 \le i \le n+1. \tag{1.17}$$

Since the $c_i^{I_i}$ are uniquely determined by $\{L_1, L_i, L_{n+2}, ..., L_{2n+1}\}$, $\frac{l_i}{l_1}$ is uniquely determined. Again the x_i are uniquely determined by $\{l_1, l_2, ..., l_{n+1}\}$, therefore the number of S-integral points $(x_0, ..., x_n) = (\frac{x_0}{l_1}, ..., \frac{x_n}{l_1})$ is

$$1. \tag{1.18}$$

If $u > n+1$, then each $\frac{l_i}{l_1}$, $1 \le i \le n+1$, is contained in one of the irreducible equations in (1.13). We may assume that we have the following irreducible equations for each $1 \le i \le n+1$.

$$-c_i \frac{l_i}{l_1} - c_{i2} \frac{l_{i2}}{l_1} - \cdots - c_{ij_i} \frac{l_{ij_i}}{l_1} = 1, \quad 1 \le i \le n+1, \ 0 \le j_i \le n-2. \tag{1.19}$$

Since no proper subsum of (1.19) vanishes, the number of solutions $(\frac{l_i}{l_1}, \frac{l_{i2}}{l_1}, ..., \frac{l_{ij_i}}{l_1})$ of (1.19) is no more than

$$(4sd!)2^{36(n-1)d!s^6}.$$

Therefore the number of $\frac{l_i}{l_1}$ is bounded by

$$(4sd!)2^{36(n-1)d!s^6}. \tag{1.20}$$

The number of S-integral points $(x_0, ..., x_n)$ in $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$, which is equal to the number of $(1, \frac{l_2}{l_1}, ..., \frac{l_{n+1}}{l_1})$ is therefore bounded by

$$(4sd!)^n 2^{36n(n-1)d!s^6}. \tag{1.21}$$

(1.12), (1.18), (1.21) imply that the number of S-integral points in $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$ is no more than

$$(4sd!)^n 2^{36n(n+1)d!s^6}. \tag{1.22}$$

2.0. Introduction—Function Fields

Let K be the function field of an irreducible projective algebraic curve C of genus g defined over an algebraically closed field k. Assume that the characteristic of k is 0. If $P \in C$ we denote by v_P the valuation of K associated to P. For elements $f_0, ..., f_n$ of K, not all zeros, we define the height as

$$h(f_0, ..., f_n) := \sum_{P\in C} -\min\{v_P(f_0), ..., v_P(f_n)\}. \tag{2.1}$$

For an element f of K we define the height as

$$h(f) := \sum_{P\in C} -\min\{0,\ v_P(f)\}. \tag{2.2}$$

Let S be a finite set of points of C. An element $f \in K$ is said to be an S-unit if $v_P(f) = 0$ for all $P \notin S$. Let $L_i = \sum_{j=0}^{n} a_{ij}X_j$, where $a_{ij} \in K$ and $1 \le i \le q$. Suppose that $q \ge 2n + 1$ and that $L_1, ..., L_q$ are in general position. When the coefficients of these linear forms are in the constant field k, we have the following theorem in [Wa]:

Theorem (Wang). *Suppose $L_1, ..., L_q$ are as stated above and with coefficients in k. Let H_i, $1 \le i \le q$, be the hyperplane corresponding to L_i. If $q \ge 2n+1$, then the height of the S-integral points of $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$ is bounded by $\frac{n(n+1)}{2}\max\{0, 2g-2+|S|\}$. Furthermore, there exist finitely many sets of $\{\eta_0,...,\eta_n\}$ in K, which can be effectively determined such that the S-integral points of $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$ are uniquely determined by $\eta_0, ..., \eta_n$.*

The proof of this theorem basically follows from the Truncated Second Main Theorem and Nochka weights. Therefore it is similar to [R-W].

For the rest of the paper, we will consider the case when the coefficients of the linear forms are non-constants. The proof is basically the same as the proof for number fields. Therefore when there is no confusion we will use the results and definitions from Section 1.0 and Section 1.1 directly.

2.1. The Main Theorem–Function Fields

We recall the following S-unit Theorem from [B-M]:

Theorem (Brownawell-Masser). *Let S be a finite set of points of C. If $f_0, ..., f_n$ are S-units and $f_0 + \cdots + f_n = 1$, then either some proper subsum of $f_0 + \cdots + f_n$ vanishes or*

$$h(f_0, ..., f_n) \le \frac{n(n+1)}{2}\max\{0,\ 2g-2+|S|\}. \tag{2.3}$$

Let $L_i = \sum_{j=0}^{n} a_{ij}X_j$ where $a_{ij} \in K$ and $1 \le i \le q$. Suppose $q \ge 2n+1$ and $L_1, ..., L_q$ are in general position. Let $I = \{i_1, ..., i_{n+2}\}$ be an index subset of $\{1, ..., q\}$ and

$$\triangle^I = \begin{pmatrix} a_{i_1,0} & \cdots & a_{i_1,n} \\ \vdots & \ddots & \vdots \\ a_{i_{n+2},0} & \cdots & a_{i_{n+2},n} \end{pmatrix}.$$

Let $\triangle^I_{i_j}$ be the determinant of the matrix obtained by deleting the j-th row from $\triangle^I$. Then by equation (1.10) we have

$$\triangle^I_{i_1} L_{i_1}(\mathbf{X}) + (-1)\triangle^I_{i_2} L_{i_2}(\mathbf{X}) + \cdots + (-1)^{n+1}\triangle_{i_{n+2}} L_{i_{n+2}}(\mathbf{X}) = 0. \tag{2.4}$$

Main Theorem (Function Fields). *Let* $L_i = \sum_{j=0}^{n} a_{ij} X_j$, *where* $a_{ij} \in K$ *and* $1 \le i \le q$. *Let* S *be a finite set of points in* C *such that all the non-zero coefficients of those linear forms and all* $\triangle^I_{i_j}$ *as we defined above are* S*-units. If* $q \ge 2n+1$ *and* $L_1, ..., L_q$ *are in general position, then the height of the* S*-integral points* $(f_0, ..., f_n)$ *of* $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$ *is bounded by*

$$(n^2 + 2n)\sum_{i=1}^{q}\sum_{j=0}^{n} h(a_{ij}) + \frac{n^2(n+1)}{2}\max\{0,\ 2g - 2 + |S|\}.$$

Furthermore, there exist finitely many sets of $\{\eta_0, ..., \eta_n\}$ *in* K, *which can be effectively determined and the number of such sets* (*up to constant factors*) *is bounded by*

$$[n(n+1)\max\{0,\ 2g-2+|S|\} + 1]^{(n+1)|S|}$$

such that the S*-integral points of* $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$ *are uniquely determined by* $\eta_0, ..., \eta_n$.

Proposition 2.1. *Let* H *be a positive integer. Then* (*up to constant factors*) *the number of* S*-units* f *in* K *with* $h(f) \le H$ *is at most*

$$(2H+1)^{|S|}. \tag{2.5}$$

Proof. Since f is an S-unit and $h(f) \le H$,

$$|v_P(f)| \le B, \text{ if } P \in S,$$
$$v_P(f) = 0, \text{ if } P \notin S.$$

Also if f, g are non-zero elements of K and $v_P(f) = v_P(g)$ for all $P \in C$, then $\frac{f}{g} \in k$. The proposition is therefore clear.

Proof of the Main Theorem. If $(x_0, ..., x_n)$ is an S-integal point in $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$, then $\frac{l_i}{l_1}$ is an S-unit, for $1 \le i \le q$, where $l_i = L_i(x_0, ..., x_n)$. Let $I = \{i_0, ..., i_n\}$ be an index subset in $\{2, ..., q\}$. By Proposition 1.3.b, there exist non-zero elements $c^I_{i_j} = \pm\frac{\triangle^I_{i_j}}{\triangle^I_1}$, $0 \le j \le n$, such that

$$L_1(\mathbf{X}) + c^I_{i_0} L_{i_0}(\mathbf{X}) + \cdots + c^I_{i_n} L_{i_n}(\mathbf{X}) = 0.$$

Therefore

$$-c^I_{i_0}\frac{l_{i_0}}{l_1} - \cdots - c^I_{i_n}\frac{l_{i_n}}{l_1} = 1. \tag{2.6}$$

If no proper subsum of (2.6) vanishes, then, by the S-unit Theorem for function fields,

$$h(c^I_{i_0}\frac{l_{i_0}}{l_1}, \ldots, c^I_{i_n}\frac{l_{i_n}}{l_1}) \le \frac{n(n+1)}{2}\max\{0,\ 2g-2+|S|\}. \tag{2.7}$$

Since $c^I_{i_0}\frac{l_{i_0}}{l_1}+\cdots+c^I_{i_n}\frac{l_{i_n}}{l_1}=-1$,

$$h(c^I_{i_j}\frac{l_{i_j}}{l_1})\leq h(c^I_{i_0}\frac{l_{i_0}}{l_1},\ldots,c^I_{i_n}\frac{l_{i_n}}{l_1}) \text{ for } 0\leq j\leq n. \tag{2.8}$$

From (2.8) and Proposition 2.1, we see the number of solutions $\frac{l_{i_j}}{l_1}$, which is equal to the number of solutions (up to constant factors) $c^I_{i_j}\frac{l_{i_j}}{l_1}$ is no more than

$$[n(n+1)\max\{0,\ 2g-2+|S|\}+1]^{|S|}. \tag{2.9}$$

Let

$$A^I=(a_{mi_j})_{0\leq m,j\leq n}, \tag{2.10}$$

where the a_{mi_j}'s are the coefficients of the linear forms. Let B^I be the inverse matrix of A^I. Then the mj-th component of B^I is equal to

$$(-1)^{m+j}\frac{1}{\det A^I}\det A^I_{jm}, \tag{2.11}$$

where A^I_{jm} is the $n\times n$ matrix obtained from A^I by deleting the j-th row and m-th column. Then

$$f_m=\sum_{j=0}^{n}(-1)^{m+j}\frac{1}{\det A^I}\det A^I_{jm}l_{i_j},\quad 0\leq m\leq n. \tag{2.12}$$

Therefore $\frac{f_j}{i_1}$ can be expressed uniquely as a linear combination of $\{\frac{l_{i_0}}{l_1},...,\frac{l_{i_n}}{l_1}\}$, and the number (up to constant factors) of possible $\{\frac{l_{i_0}}{l_1},...,\frac{l_{i_n}}{l_1}\}$, which is equal to the number (up to constant factors) of $\{c^I_{i_0}\frac{l_{i_0}}{l_1},...,c^I_{i_n}\frac{l_{i_n}}{l_1}\}$, is no more than

$$[n(n+1)\max\{0,\ 2g-2+|S|\}+1]^{(n+1)|S|}. \tag{2.13}$$

Furthermore, by (2.12) we have

$$\begin{aligned} h(f_0,...,f_n)&=h(\frac{\det A^I}{l_1}f_0,...,\frac{\det A^I}{l_1}f_n)\\ &=h(\sum_{j=0}^{n}(-1)^j\det A^I_{j0}\frac{l_{i_j}}{l_1},...,\sum_{j=0}^{n}(-1)^{j+n}\det A^I_{jn}\frac{l_{i_j}}{l_1})\\ &\leq h(\det A^I_{00},...,\det A^I_{ji},...,\det A^I_{nn})\\ &\quad +h(\frac{1}{c_{i_0}},...,\frac{1}{c_{i_n}})+h(c_{i_0}\frac{l_{i_0}}{l_1},...,c_{i_n}\frac{l_{i_n}}{l_1}). \end{aligned} \tag{2.14}$$

The determinant of A^I_{jm} can be written down explicitly,

$$\det A^I_{jm}=\sum_{\sigma_{jm}}\epsilon(\sigma_{jm})\prod_{\substack{u\neq j\\ 1\leq w\leq n}}a_{\sigma_{jm}(w),i_u}, \tag{2.15}$$

where the sum is taken over all permutations σ_{jm} of $\{0,1,...,j-1,j+1,...,n\}$, and $\epsilon(\sigma_{jm})$ is the sign of the permutation. (See [La 2], XIII Proposition 4.6.) Therefore

$$v_P(\det A^I_{jm}) \geq \min_{\sigma_{jm}}\{v_P(\prod_{\substack{u\neq j\\ 1\leq w\leq n}} a_{\sigma_{jm}(w),i_u})\}$$

$$(2.16)\qquad \geq n \min_{\substack{1\leq i\leq q\\ 0\leq j\leq n}} \min\{0,\ v_P(a_{ij})\}$$

$$\geq n\sum_{i=1}^{q}\sum_{j=0}^{n}\min\{0,\ v_P(a_{ij})\}.$$

Therefore

$$(2.17)\qquad h(\det A^I_{00},...,\det A^I_{ji},...,\det A^I_{nn}) \leq n\sum_{i=1}^{q}\sum_{j=0}^{n}h(a_{ij}).$$

Since $c^I_{i_j} = \frac{\triangle^I_{i_j}}{\triangle^I_1}$ up to sign, using the same argument, we can obtain

$$h(\frac{1}{c_{i_0}},...,\frac{1}{c_{i_n}}) = h(\frac{1}{\triangle^I_{i_0}},...,\frac{1}{\triangle^I_{i_n}})$$

$$(2.18)\qquad = h(\prod_{j\neq 0}\triangle^I_{i_j},...,\prod_{j\neq n}\triangle^I_{i_j})$$

$$\leq n(n+1)\sum_{i=1}^{q}\sum_{j=0}^{n}h(a_{ij}).$$

(2.7), (2.14), (2.17), and (2.18) imply that

$$(2.19)\quad h(f_0,...,f_n) \leq (n^2+2n)\sum_{i=1}^{q}\sum_{j=0}^{n}h(a_{ij}) + \frac{n(n+1)}{2}\max\{0,\ 2g-2+|S|\}.$$

The case is therefore complete.

It then suffices to consider the case that for any index subset $I=\{i_0,...,i_n\}$ in $\{2,...,q\}$, some proper subsums of (2.6) vanish. By Proposition 1.1, for any $\{L_{i_0},...,L_{i_n}\}$ there exist some irreducible equations in the following form:

$$(2.20)\qquad -c^I_{j_0}\frac{l_{j_0}}{l_1} - \cdots - c^I_{j_k}\frac{l_{j_k}}{l_1} = 1,$$

where $\{i_0,...,i_n\}$ is an index subset of $\{2,...,q\}$ and $\{j_0,...,j_k\}$ is an index subset of $\{i_0,...,i_n\}$. There are only finitely many such irreducible equations in total. Let $T_{\mathbf{f}}$ be the subset of $\{L_1,...,L_q\}$ with the following property:

(2.21)

$L_i \in T_{\mathbf{f}}$ if and only if $\frac{l_i}{l_1}$ appears in one of the irreducible equations in (2.20).

Rearranging the order of the linear forms, we may assume

$$T_{\mathbf{f}} = \{L_1,...,L_u\}.$$

As in the proof of the number field case, we only have to consider when $u \geq n+1$. If $u = n+1$, the same proof goes through and

$$\frac{l_i}{l_1} = -\frac{1}{c_i^{I_i}}, \quad 2 \leq i \leq n+1.$$

Therefore in this case the number of S-integral points $(f_0, ..., f_n)$ in $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$ is

$$1. \tag{2.22}$$

Moreover, together with (2.14), (2.17) and (2.18) we get

$$h(f_0, ..., f_n) \leq (n^2 + 2n) \sum_{i=1}^{q} \sum_{j=0}^{n} h(a_{ij}). \tag{2.23}$$

If $u > n+1$, then each $\frac{l_i}{l_1}$, $1 \leq i \leq n+1$, is contained in one of the irreducible equations in (2.20). We may assume that we have the following irreducible equations for each $2 \leq i \leq n+1$.

$$-c_i \frac{l_i}{l_1} - c_{i2} \frac{l_{i2}}{l_1} - \cdots - c_{ij_i} \frac{l_{ij_i}}{l_1} = 1, \quad 1 \leq i \leq n+1, \ 2 \leq j_i \leq n-1. \tag{2.24}$$

Since no proper subsum of (2.24) vanishes,

$$h(c_i \frac{l_i}{l_1}) \leq \frac{n(n-1)}{2} \max\{0, \ 2g - 2 + |S|\}. \tag{2.25}$$

Therefore the number of $\frac{l_i}{l_1}$ (up to constant factors) is bounded by

$$[n(n-1)\max\{0, \ 2g-2+|S|\} + 1]^{|S|}.$$

Therefore the number of S-integral points $(f_0, ..., f_n)$ in $\mathbb{P}^n(K) - \{\bigcup_{i=1}^q H_i\}$, which is equal to the number of $(1, \frac{l_2}{l_1}, ..., \frac{l_{n+1}}{l_1})$, is bounded by

$$[n(n-1)\max\{0, \ 2g-2+|S|\} + 1]^{n|S|}. \tag{2.26}$$

Furthermore, if $I = \{1, 2, ..., n+1\}$, by (2.14), (2.17), (2.18) and (2.25)

$$\begin{aligned} & h(f_0, ..., f_n) \\ & \leq (n^2 + 2n) \sum_{i=1}^{q} \sum_{j=0}^{n} h(a_{ij}) + h(1, c_2 \frac{l_2}{l_1}, ..., c_{n+1} \frac{l_{n+1}}{l_1}) \\ & \leq (n^2 + 2n) \sum_{i=1}^{q} \sum_{j=0}^{n} h(a_{ij}) + \sum_{i=2}^{n+1} (h(c_i \frac{l_i}{l_1})) \\ & \leq (n^2 + 2n) \sum_{i=1}^{q} \sum_{j=0}^{n} h(a_{ij}) + \frac{n^2(n-1)}{2} \max\{0, \ 2g-2+|S|\}. \end{aligned} \tag{2.27}$$

By (2.19), (2.23), and (2.27),

$$h(f_0, ..., f_n) \leq (n^2 + 2n) \sum_{i=1}^{q} \sum_{j=0}^{n} h(a_{ij}) + \frac{n^2(n+1)}{2} \max\{0, \ 2g-2+|S|\}. \tag{2.28}$$

From (2.12), (2.13), (2.22), and (2.26), we see that we can find $\{l_{i_0}, ..., l_{i_n}\}$ effectively such that the f_i can be expressed uniquely as a linear combination of this set, and the number (up to constant factors) of the sets $\{l_{i_0}, ..., l_{i_n}\}$ is bounded by

$$[n(n+1)\max\{0,\ 2g-2+|S|\}+1]^{(n+1)|S|}. \tag{2.29}$$

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